Lecture 6: Block Theory, Primitive Idempotents, and Defect Groups

Goal: Understand the decomposition of the group algebra into blocks, the role of primitive central idempotents, the structure of blocks as indecomposable summands of the group algebra, and the significance of defect groups in controlling modular representation theory.

1. Motivation and Overview of Blocks

In modular representation theory, the group algebra $F[G]$, where $\mathrm{char}(F) = p \mid |G|$, is not semisimple. Nevertheless, it decomposes into *indecomposable* two-sided ideals called *blocks*.

Definition 6.1 (Block). A *block* of $F[G]$ is an indecomposable two-sided ideal of $F[G]$ which is a direct summand as a ring:

$$F[G] = B_1 \oplus B_2 \oplus \cdots \oplus B_r,$$

where each B_i is a block.

Proposition 6.2. Each block B_i corresponds to a central primitive idempotent $e_i \in Z(F[G])$, such that:

$$1 = e_1 + \cdots + e_r, \quad e_i e_j = \delta_{ij} e_i.$$

2. Primitive Central Idempotents

Definition 6.3. An idempotent $e \in A$ is called:

- *central* if $e \in Z(A)$,
- *primitive* if $e \neq 0$ and cannot be written as $e = e_1 + e_2$ for orthogonal idempotents $e_1, e_2 \in Z(A)$.

Theorem 6.4. The central primitive idempotents of $F[G]$ correspond bijectively to the blocks. Each irreducible representation (modular or ordinary) belongs to exactly one block.

Definition 6.5. The *block idempotent* e_B is the central primitive idempotent corresponding to block B, so that:

$$F[G] = \bigoplus_B F[G]e_B.$$

3. Block Components of Characters

Definition 6.6. Let $\chi \in \mathrm{Irr}(G)$ be an ordinary irreducible character. Then χ belongs to the block B if its projective cover reduces mod p to a composition series in B.

Equivalently, $\chi(e_B) \neq 0$.

Proposition 6.7. Each irreducible Brauer character lies entirely in a single block. The decomposition matrix is block-diagonalized accordingly.

4. Defect Groups

Definition 6.8. Let B be a block of $F[G]$. A *defect group* D of B is a minimal p-subgroup $D \leq G$ such that B is a summand of the group algebra $F[N_G(D)]$, where $N_G(D)$ is the normalizer of D.

Theorem 6.9. Blocks with trivial defect (i.e., defect group $\{1\}$) are called *blocks of defect zero* and correspond to irreducible projective modules (i.e., the character lifts and reduces irreducibly).

Corollary 6.10. If an irreducible character χ satisfies $\gcd(|G|/\chi(1), p) = 1$, then it lies in a block of defect zero.

5. Examples

Example 6.11 (Blocks in S_3 over $\mathbb{F}_3$).

- $|S_3| = 6$, $p = 3$
- Conjugacy classes: $C_1 = \{1\}$, $C_2 = \{(12), (13), (23)\}$, $C_3 = \{(123), (132)\}$
- C_3 is p-singular, so there are only two p-regular classes
- Hence, two irreducible Brauer characters $\rightarrow$ two blocks

6. Counterexamples

Counterexample 6.12. The decomposition matrix is not necessarily block-diagonal unless the basis is ordered according to block membership.

Counterexample 6.13. In characteristic zero, every irreducible character lies in its own block. In modular representation theory, blocks may contain multiple irreducible characters.

7. Summary

In this lecture we learned:

- The group algebra in characteristic p decomposes into blocks
- Blocks correspond to central primitive idempotents
- Each irreducible (ordinary or modular) character lies in a unique block
- Defect groups determine the "size" and complexity of a block

Coming Up in Lecture 7: We'll look at *projective modules*, understand the role of *projective covers* in modular representation theory, and link them with Brauer characters and decomposition numbers.